Inclusive τ lepton hadronic decay in vector and axial–vector channels within dispersive approach to QCD

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Abstract. The dispersive approach to QCD, which properly embodies the intrinsically nonperturbative constraints originating in the kinematic restrictions on relevant physical processes and extends the applicability range of perturbation theory towards the infrared domain, is briefly overviewed. The study of OPAL (update 2012) and ALEPH (update 2014) experimental data on inclusive τ lepton hadronic decay in vector and axial–vector channels within dispersive approach is presented.

Keywords: nonperturbative methods, low–energy QCD, dispersion relations, τ lepton hadronic decay
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The theoretical particle physics widely employs the methods based on dispersion relations. In particular, such methods have proved to be efficient in the extension of the range of applicability of chiral perturbation theory [1, 2], assessment of the hadronic light–by–light scattering [3], precise determination of parameters of resonances [4], and many other issues.

The dispersion relations render the kinematic restrictions on pertinent physical processes into the mathematical form and impose stringent nonperturbative constraints on relevant quantities, such as the hadronic vacuum polarization function $\Pi(q^2)$. These constraints have been properly embodied within dispersive approach to QCD[1] [7, 8], which provides unified integral representations for $\Pi(q^2)$, related function $R(s)$, which is identified with the so–called R–ratio of electron–positron annihilation into hadrons, and Adler function $D(Q^2)$:

$$\Delta\Pi(q^2, q_0^2) = \Delta\Pi^{(0)}(q^2, q_0^2) + \int_{m^2}^{\infty} \rho(\sigma) \ln\left(\frac{\sigma - q^2}{\sigma - q_0^2}\,\frac{m^2 - q_0^2}{m^2 - q^2}\right) \frac{d\,\sigma}{\sigma}, \tag{1}$$

$$R(s) = R^{(0)}(s) + \theta(s - m^2) \int_{s}^{\infty} \rho(\sigma) \frac{d\,\sigma}{\sigma}, \tag{2}$$

$$D(Q^2) = D^{(0)}(Q^2) + \frac{Q^2}{Q^2 + m^2} \int_{m^2}^{\infty} \rho(\sigma) \frac{\sigma - m^2}{\sigma + Q^2} \frac{d\,\sigma}{\sigma}. \tag{3}$$

In these equations m denotes the value of hadronic production threshold, $\rho(\sigma)$ is the spectral density

$$\rho(\sigma) = \frac{1}{2\pi i} \frac{d}{d \ln\sigma} \lim_{\varepsilon \to 0_+} \Big[p(\sigma - i\varepsilon) - p(\sigma + i\varepsilon)\Big] = -\frac{d\,r(\sigma)}{d \ln\sigma} = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \Big[d(-\sigma - i\varepsilon) - d(-\sigma + i\varepsilon)\Big], \tag{4}$$

$\Delta\Pi(q^2, q_0^2) = \Pi(q^2) - \Pi(q_0^2)$ stands for the subtracted hadronic vacuum polarization function, whereas $p(q^2)$, $r(s)$, and $d(Q^2)$ denote the strong corrections to the functions $\Pi(q^2)$, $R(s)$, and $D(Q^2)$, respectively. The derivation of integral representations (1)–(3) employs only the kinematic restrictions on the relevant physical processes, the asymptotic ultraviolet behavior of the hadronic vacuum polarization function, and requires neither additional approximations nor phenomenological assumptions, see Refs. [7, 8].

The common prefactor $N_{\text{c}} \sum_{f=1}^{n_{\text{f}}} Q_f^2$ is omitted throughout the paper, where $N_{\text{c}} = 3$ is the number of colors, Q_f stands for the electric charge of f–th quark, and n_{f} is the number of active flavors. In Eqs. (1)–(3) $Q^2 = -q^2 > 0$ and $s = q^2 > 0$ denote the spacelike and timelike kinematic variables, respectively, and $\theta(x)$ is the unit step–function [$\theta(x) = 1$ if $x \geq 0$

[1] Its preliminary formulation was discussed in Refs. [5, 6].

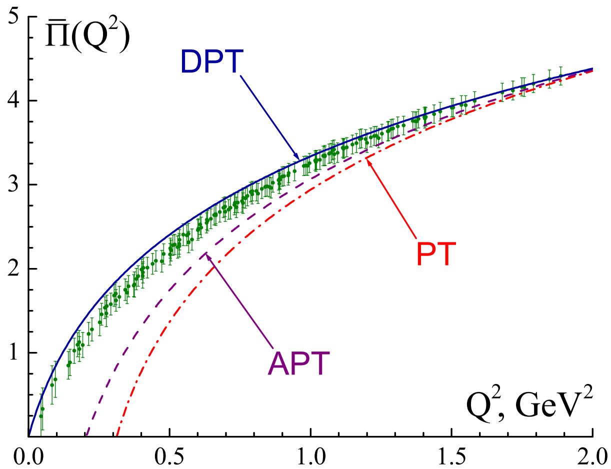

FIGURE 1. Comparison of the hadronic vacuum polarization function $\bar{\Pi}(Q^2) = \Delta\Pi(0, -Q^2)$ with relevant lattice simulation data [14], see Ref. [15] for the details.

and $\theta(x) = 0$ otherwise]. The leading–order terms in Eqs. (1)–(3) read

$$\Delta\Pi^{(0)}(q^2, q_0^2) = 2\,\frac{\varphi - \tan\varphi}{\tan^3\varphi} - 2\,\frac{\varphi_0 - \tan\varphi_0}{\tan^3\varphi_0}, \tag{5}$$

$$R^{(0)}(s) = \theta(s - m^2)\left(1 - \frac{m^2}{s}\right)^{3/2}, \tag{6}$$

$$D^{(0)}(Q^2) = 1 + \frac{3}{\xi}\left[1 - \sqrt{1+\xi^{-1}}\,\sinh^{-1}\!\left(\xi^{1/2}\right)\right], \tag{7}$$

where $\sin^2\varphi = q^2/m^2$, $\sin^2\varphi_0 = q_0^2/m^2$, and $\xi = Q^2/m^2$, see papers [8, 9, 10] and references therein for the details.

There is still no unambiguous method to restore the complete expression for the spectral density $\rho(\sigma)$ (4) (discussion of this issue can be found in, e.g., Refs. [9, 10, 11]). Nonetheless, the perturbative contribution to $\rho(\sigma)$ can be calculated by making use of the perturbative expression for either of the strong corrections to the functions on hand (see, e.g., Refs. [12, 13]):

$$\rho_{\text{pert}}(\sigma) = \frac{1}{\pi}\frac{d}{d\ln\sigma}\,\mathrm{Im}\lim_{\varepsilon\to 0_+} p_{\text{pert}}(\sigma - i\varepsilon) = -\frac{d\,r_{\text{pert}}(\sigma)}{d\ln\sigma} = \frac{1}{\pi}\,\mathrm{Im}\lim_{\varepsilon\to 0_+} d_{\text{pert}}(-\sigma - i\varepsilon). \tag{8}$$

In this paper the model [8] for the spectral density will be employed:

$$\rho(\sigma) = \frac{4}{\beta_0}\frac{1}{\ln^2(\sigma/\Lambda^2) + \pi^2} + \frac{\Lambda^2}{\sigma}, \tag{9}$$

where $\beta_0 = 11 - 2n_{\text{f}}/3$ and Λ denotes the QCD scale parameter. The first term on the right–hand side of Eq. (9) is the one–loop perturbative contribution, whereas the second term represents intrinsically nonperturbative part of the spectral density, see paper [8] and references therein for the details.

It is worthwhile to mention also that in the massless limit ($m = 0$) for the case of perturbative spectral function $[\rho(\sigma) = \mathrm{Im}\, d_{\text{pert}}(-\sigma - i0_+)/\pi]$ two equations (2) and (3) become identical to those of the analytic perturbation theory (APT) [16] (see also Refs. [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28]). However, it is essential to keep the value of hadronic production threshold nonvanishing, since the massless limit loses some of the substantial nonperturbative constraints, which relevant dispersion relations impose on the functions on hand, see Refs. [7, 8, 15, 29].

The dispersively improved perturbation theory (DPT) [7, 8] extends the applicability range of perturbative approach towards the infrared domain. In particular, the Adler function[2] (3) conforms with relevant experimental prediction

[2] The studies of Adler function within other approaches can be found in Refs. [30, 31, 32, 33, 34, 35, 36, 37].

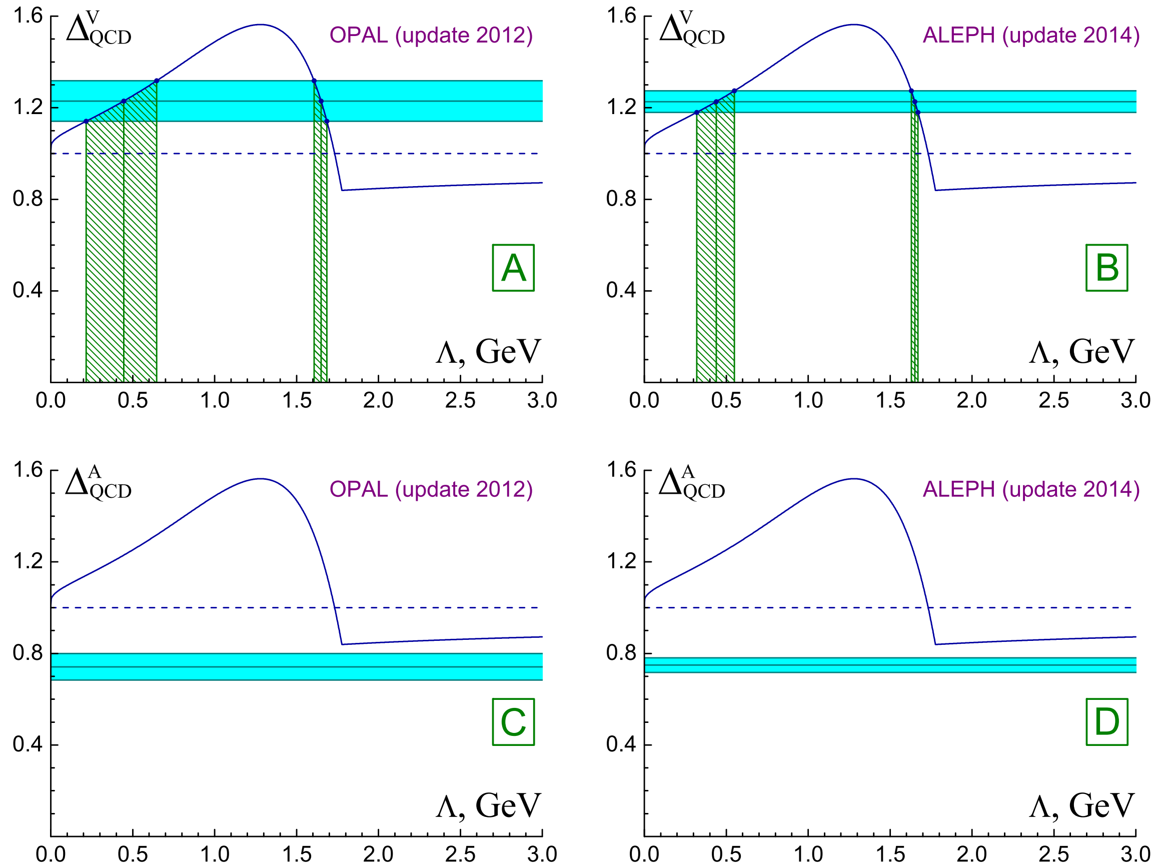

FIGURE 2. Comparison of the perturbative expression $\Delta^{\text{V/A}}_{\text{pert}}$ (12) (solid curves) with relevant experimental data (horizontal shaded bands). Vertical dashed bands denote solutions for the QCD scale parameter Λ. The plots A, C and B, D correspond to experimental data [45] and [46], respectively.

in the entire energy range [7, 29, 38] and the hadronic vacuum polarization function (1) agrees with pertinent lattice simulation data [15], see Fig. 1. Furthermore, the representations (1)–(3) conform with the results obtained in Ref. [39] as well as in Ref. [40]. Additionally, the respective hadronic contributions to the muon anomalous magnetic moment and to the shift of the electromagnetic fine structure constant at the scale of Z boson mass evaluated in the framework of DPT proved to be in a good agreement with recent estimations of these quantities [15]. All this testifies to the efficiency of dispersive approach [7, 8] in the studies of nonperturbative aspects of the strong interaction.

The study of the inclusive τ lepton hadronic decay represents a particular interest, since this process probes the low–energy hadron dynamics. Specifically, the theoretical expression for the relevant experimentally measurable quantity reads

$$R^{J=1}_{\tau,\text{V/A}} = \frac{N_{\text{c}}}{2}\,|V_{\text{ud}}|^2 S_{\text{EW}} \left(\Delta^{\text{V/A}}_{\text{QCD}} + \delta'_{\text{EW}}\right). \tag{10}$$

In this equation $|V_{\text{ud}}| = 0.97425 \pm 0.00022$ is Cabibbo–Kobayashi–Maskawa matrix element [41], $\delta'_{\text{EW}} = 0.0010$ and $S_{\text{EW}} = 1.0194 \pm 0.0050$ denote the electroweak corrections [42], and

$$\Delta^{\text{V/A}}_{\text{QCD}} = \frac{2}{\pi}\int_{m^2_{\text{V/A}}}^{M_\tau^2} \left(1-\frac{s}{M_\tau^2}\right)^2 \left(1+2\frac{s}{M_\tau^2}\right) \operatorname{Im}\Pi^{\text{V/A}}(s+i0_+)\,\frac{ds}{M_\tau^2} \tag{11}$$

stands for the hadronic contribution, see Refs. [43, 44]. In Eq. (11) $M_\tau \simeq 1.777\,\text{GeV}$ [41] is the mass of τ lepton, whereas $m_{\text{V/A}}$ denotes the total mass of the lightest allowed hadronic decay mode of τ lepton in the corresponding channel.

It is worthwhile to mention that the perturbative description of the inclusive τ lepton hadronic decay completely leaves out the effects due to the nonvanishing hadronic production threshold. Moreover, the perturbative approach

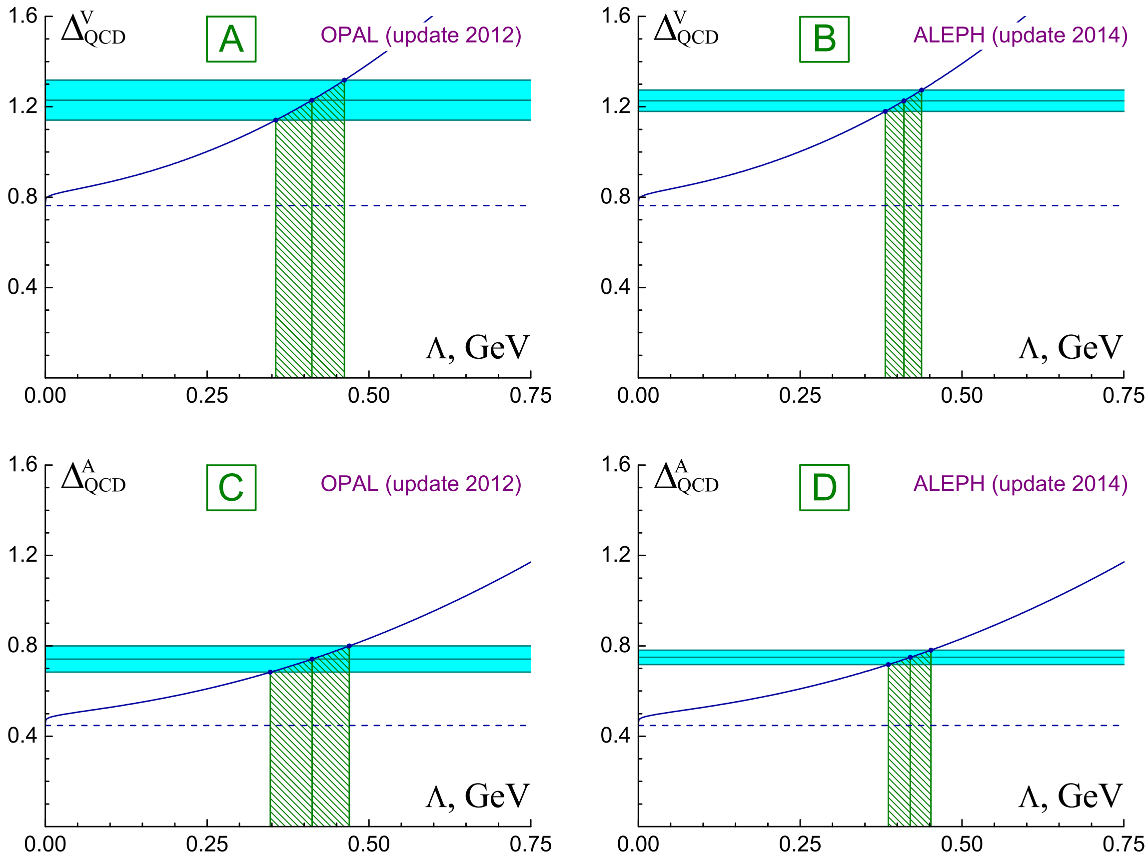

FIGURE 3. Comparison of the expression $\Delta^{\rm V/A}_{\rm QCD}$ (14) (solid curves) with relevant experimental data (horizontal shaded bands). Vertical dashed bands denote solutions for the QCD scale parameter Λ. The plots A, C and B, D correspond to experimental data [45] and [46], respectively.

suffers from its inherent difficulties, such as the infrared unphysical singularities. These facts eventually result in the identity of the perturbative predictions for functions (11) in vector and axial–vector channels (i.e., $\Delta^{\rm V}_{\rm pert} \equiv \Delta^{\rm A}_{\rm pert}$), that contradicts experimental data. In particular, within perturbative approach the expression (11) acquires the form (in what follows the one–loop level with $n_{\rm f} = 3$ active flavors is assumed)

$$\Delta^{\rm V/A}_{\rm pert} = 1 + \frac{4}{\beta_0} \int_0^{\pi} \frac{\lambda A_1(\theta) + \theta A_2(\theta)}{\pi(\lambda^2 + \theta^2)} \, d\theta, \tag{12}$$

where $\lambda = \ln\bigl(M_\tau^2/\Lambda^2\bigr)$, and

$$A_1(\theta) = 1 + 2\cos(\theta) - 2\cos(3\theta) - \cos(4\theta), \qquad A_2(\theta) = 2\sin(\theta) - 2\sin(3\theta) - \sin(4\theta), \tag{13}$$

see Refs. [8, 47]. Furthermore, the perturbative approach is incapable of describing the experimental data on the inclusive semileptonic branching ratio in axial–vector channel, see Fig. 2 and Table 1. It is worth noting also that for vector channel perturbative approach returns two equally justified solutions for the QCD scale parameter Λ, one of which is commonly discarded, see paper [8] and references therein for the details.

The inclusive τ lepton hadronic decay was also studied within analytic perturbation theory and a number of its modifications [33, 48, 49]. However, these papers basically deal either with the total sum of vector and axial–vector terms (10) or with the vector term only. Additionally, APT disregards valuable effects due to nonvanishing hadronic production threshold and, similarly to perturbative approach, yields identical predictions for functions (11) in vector and axial–vector channels. For the vector channel APT returns a rather large value for the QCD scale parameter ($\Lambda \simeq 900\,$MeV). As for the axial–vector channel, the APT fails to describe the experimental data on the inclusive τ lepton hadronic decay, since for any value of Λ the APT expression for function (11) exceeds its experimental measurement, see also Ref. [9].

TABLE 1. Values of the QCD scale parameter Λ [MeV] obtained within perturbative and dispersive approaches from OPAL [45] and ALEPH [46] experimental data on inclusive τ lepton hadronic decay (one–loop level, $n_{\mathrm{f}} = 3$ active flavors), see Refs. [8, 47].

	Perturbative approach		Dispersive approach	
	OPAL [45] (update 2012)	ALEPH [46] (update 2014)	OPAL [45] (update 2012)	ALEPH [46] (update 2014)
Vector channel	445^{+201}_{-230}	439^{+110}_{-119}	409 ± 53	409 ± 28
Axial–vector channel	no solution		409 ± 61	419 ± 33

The dispersive approach to QCD (contrary to perturbative and analytic approaches) properly accounts for the effects due to nonvanishing hadronic production threshold. The hadronic contribution (11) to the inclusive semileptonic branching ratio within dispersive approach can eventually be represented as

$$\Delta_{\mathrm{QCD}}^{\mathrm{V/A}} = 3 g_1\!\left(\frac{\chi_{\mathrm{V/A}}}{2}\right) \sqrt{1-\chi_{\mathrm{V/A}}} - 3 g_2\!\left(\frac{\chi_{\mathrm{V/A}}}{4}\right) \ln\!\left(\sqrt{\chi_{\mathrm{V/A}}^{-1}} + \sqrt{\chi_{\mathrm{V/A}}^{-1} - 1}\right) + \int_{m_{\mathrm{V/A}}^2}^{\infty} G\!\left(\frac{\sigma}{M_\tau^2}\right) \rho(\sigma)\, \frac{d\sigma}{\sigma}\,, \tag{14}$$

where $G(x) = g(x)\,\theta(1-x) + g(1)\,\theta(x-1) - g(\chi_{\mathrm{V/A}})$, $g(x) = x(2-2x^2+x^3)$, $\chi_{\mathrm{V/A}} = m_{\mathrm{V/A}}^2/M_\tau^2$, $m_{\mathrm{V}}^2 \simeq 0.075\,\mathrm{GeV}^2$, $m_{\mathrm{A}}^2 \simeq 0.288\,\mathrm{GeV}^2$, spectral density $\rho(\sigma)$ is specified in Eq. (9), and

$$g_1(x) = \frac{1}{3} + 4x - \frac{5}{6}x^2 + \frac{1}{2}x^3, \qquad g_2(x) = 8x(1+2x^2-2x^3), \tag{15}$$

see papers [8, 9, 10, 47] and references therein. The comparison of Eq. (14) with OPAL (update 2012, Ref. [45]) and ALEPH (update 2014, Ref. [46]) experimental data is presented in Fig. 3 and the respective values of the QCD scale parameter Λ are given in Table 1. As one may infer from Fig. 3, the dispersive approach is capable of describing the experimental data [45, 46] on inclusive τ lepton hadronic decay in vector and axial–vector channels. The obtained values of the QCD scale parameter Λ appear to be nearly identical in both channels, that testifies to the self–consistency of the approach on hand.

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